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On the Incoherent Scattering of an Acoustic or Electromagnetic Wave

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A random configuration of objects in space, or a stochastically rough boundary, is considered to scatter an incident acoustic or electromagnetic wave having harmonic time dependence $e^{i\omega t}$. In the case of a stochastic surface, Beckmann has compared the Kirchhoff solution with his approach, which employs random walk. The latter approach is used to demonstrate the Rayleigh-distributed amplitude of a field scattered by a very rough surface. This demonstration requires the conjecture that large standard deviations in the random phases of the scattered elementary waves result in an incoherent scattered field. Beckmann's conjecture has not been rigorously proven. However, in this paper, incoherence of the scattered field and broad distributions, over many cycles, in the phases of the elementary waves are both shown to be implied by a third condition, which is defined. Furthermore, the random phase of an incoherent field is shown to be statistically independent of its amplitude and uniformly distributed on a 2π-rad interval.

KEY WORDS: Propagation; acoustics; electromagnetic waves; scattering; incoherence; random walk; uncertainty principle.

1. INTRODUCTION

The total acoustic or electromagnetic-wave field scattered by a rough surface or by any configuration of scatterers, at a given point in space or in a given direction, may be considered as a sum of elementary waves in mutual phase interference.[4] The incident field has harmonic time dependence $e^{i\omega t}$, where ω is the angular frequency, and is considered to originate from a common point or a common direction.

In Fig. 1(A), a group of objects is illuminated by a point source. Each scatterer reradiates the incident wave with a changed amplitude and phase, both of which are complicated functions of the position, shape, and orientation of the scatterer, as well as of its acoustical or electrical properties.[1] Letting P_j be the phase change at the jth scatterer, $j = 1, 2,\ldots$, and denoting its distances from the source and receiving

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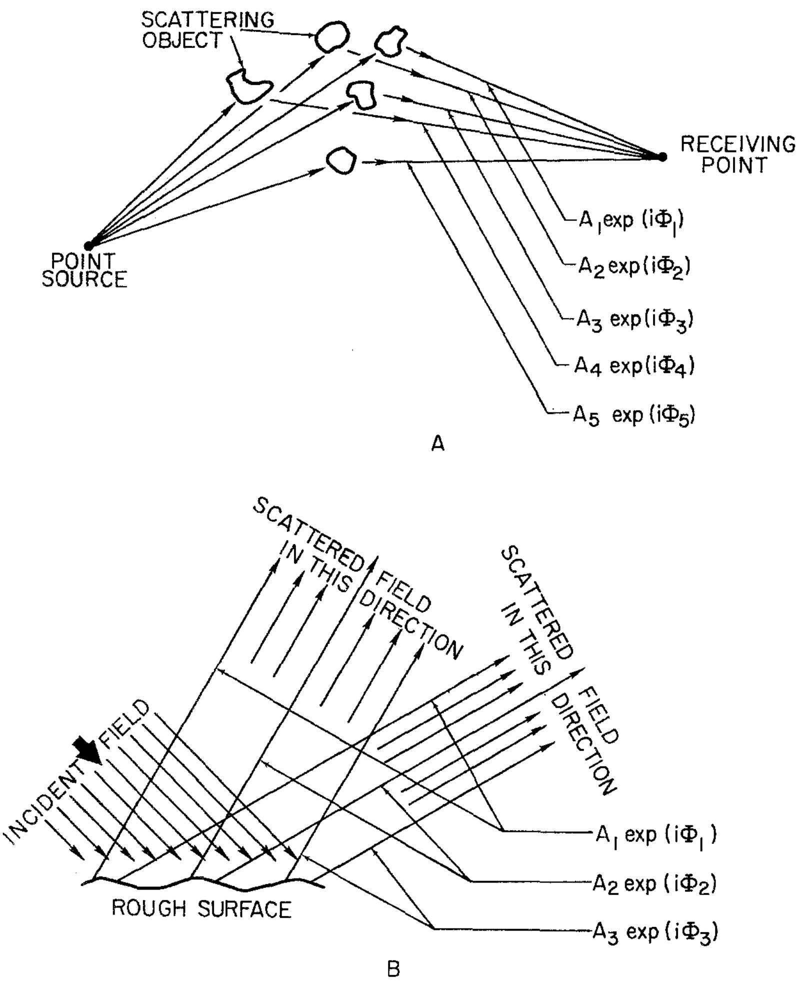

Fig. 1. (A) Incident wave-field from common point-source scattered by a configuration of objects. (B) Incident plane-wave field from common direction scattered by a rough surface.

points by d_{sj} and d_{rj}, respectively, the received phase of the scattered wave due to the jth scatterer is given by[1]

$$\Phi_j = P_j + (2\pi/\lambda)(d_{sj} + d_{rj}) \tag{1}$$

where the term ωt for the phase of the source is suppressed, and λ is the wavelength of the incident radiation. The received amplitude of this scattered wave is denoted by A_j, and is always taken to be nonnegative. Therefore, the amplitude A and phase Φ of the total scattered field at the receiving point are determined by the sum of complex numbers[1]

$$Ae^{i\Phi} = \sum_{j=1}^{M} A_j \exp(i\Phi_j), \qquad A \geqslant 0, \quad -\pi \leqslant \Phi < \pi \tag{2}$$

where M denotes the total number of scattering objects contributing significantly to the total field at the receiving point.

In Fig. 1(B), a rough surface is illuminated by a plane-wave field. The wavelength of the incident radiation is considered to be small with respect to the amplitude of the surface roughness. Then, to a first approximation, the contributions to the scattered field in a particular direction originate from those segments of the surface whose slopes are favorable to locally specular reflections[5] in that direction. In addition, the radii of curvature of these segments must be large enough with respect to the radiation's wavelength so that most of the energy there is reflected in the locally specular direction.[5] The phase of each elementary wave emitted from a corresponding surface segment can be determined from an equation similar to Eq. (1),[6] and generally depends on the position and slope of the segment as well as on the acoustical or electrical properties of the surface. Therefore, the amplitude A and phase Φ of the resultant scattered field in a particular direction are given by Eq. (2). It will be assumed in this paper that the summation in Eq. (2) is finite or can be truncated with little error after some number M of terms.

If the array of scatterers is random, or if the surface is stochastic, then the vectoral terms in Eq. (2) are random quantities, and the summation becomes a random walk. If the amplitude of the boundary roughness, or the uncertainty in position of each of the scatterers, is large with respect to the radiation's wavelength, and if the grazing angle of the incident field upon the scattering region is not too small, then each random phase Φ_j is observed to be distributed over many cycles.[1,6] [Note Eq. (1) for small λ.] Owing to the periodicity of the function $\exp(i\Phi_j)$, the sum in Eq. (2) depends only on each *primary phase* $\hat{\Phi}_j$, which is the *actual phase* Φ_j mapped into some interval of length 2π radians by adding or subtracting the necessary multiple of 2π from each outcome.

In mathematical terms, the randomly scattered field is said to be incoherent if[9]

$$E(A^2) = \sum_{j=1}^{M} E(A_j^2) \tag{3}$$

where E denotes mathematical expectation, and where it is noted that the intensity of a wave is proportional to the square of its amplitude. Observing that $\exp(i\Phi_j) = \exp(i\hat{\Phi}_j)$, it can be shown, using Eq. (2), that

$$E(A^2) = \sum_{j=1}^{M} E(A_j^2) + 2E(T) \tag{4}$$

where T is the sum of the "cross" terms

$$T = \sum_{k \neq j}\sum A_j A_k (\cos\hat{\Phi}_j \cos\hat{\Phi}_k + \sin\hat{\Phi}_j \sin\hat{\Phi}_k) \tag{5}$$

If the primary phases are independent random variables uniformly distributed on their 2π-rad intervals, then $E(T)$ vanishes, as observed in ref. 6, when the A_j are assumed to be constants. Then, Eq. (4) reduces to the incoherence condition, Eq. (3).

However, if random A_j are considered, then the additional stipulation that each A_j be independent of every primary phase $\hat{\Phi}_k$ is observed to be needed in order that $E(T)$ vanish. Therefore, in this paper, *incoherence* is defined to satisfy the following three conditions:

(*i*) Each primary phase $\hat{\Phi}_j$, $1 \leqslant j \leqslant M$, is uniformly distributed on its 2π-rad interval.

(*ii*) The primary phases are totally independent random variables.

(*iii*) The random vectors $(A_1, ..., A_M)$ and $(\hat{\Phi}_1, ..., \hat{\Phi}_M)$ are independent.

If the scattered field is not only incoherent, but also satisfies the conditions:

(*iv*) the random amplitudes $A_1, ..., A_M$ are totally independent;

(*v*) the standard deviations of the A_j are all similar in magnitude;

(*vi*) M is large;

then the resultant amplitude A of the scattered field can be shown to be nearly Rayleigh-distributed.[2] Furthermore, the resultant phase Φ is observed to be independent of A and uniformly distributed on the interval $(-\pi, \pi)$.[2] These conclusions follow directly by applying the central limit theorem to the real and imaginary components of Eq. (2).

The model in Fig. 1(A) has been applied to the propagation of vhf and uhf radio signals beyond the horizon by their scattering in the ionospheric and tropospheric layers of the atmosphere.[1] The index of refraction in these layers varies randomly from point to point as well as with time. Therefore, as a first approximation, these layers are partitioned into "blobs," i.e., inhomogeneities of uniform refractive indexes, which differ from that of the atmosphere into which these scatterers are emersed. These blobs are imagined to vary randomly in position (as well as in shape and orientation). Within a layer, the volume which is illuminated by the transmitting antenna and which contributes to the field at the receiving point is usually large enough to contain many such blobs. Therefore, condition (*vi*) is satisfied. If, for the most part, the random motions of the blobs are statistically independent, then conditions (*ii*) and (*iv*) are satisfied. If the illuminated volume is statistically homogeneous, then condition (*v*) is satisfied. In practice, the signals received from ionospheric or tropospheric scatter are found quite often to be Rayleigh-distributed.[1] Therefore, it seems plausible that somehow the remaining conditions, (*i*) and (*iii*), are also satisfied.

An analysis of Eq. (1) shows that the phase of the arrival from each blob must have a standard deviation much larger than 2π radians.[1] Beckmann indicates heuristically, though not rigorously, that actual phases which are broadly distributed over many cycles yield primary phases that are uniformly distributed on their 2π-rad intervals.[3] If this proposition were proven, then condition (*i*) would be satisfied. Condition (*iii*) has not been considered.

Short-wave radiation scattered from a rough boundary is also found to be often Rayleigh-distributed. The model in Fig. 1(B) has been applied to this problem,[7] and a discussion similar to the preceding one can be presented.

The problem here is to derive a relationship among conditions (*i*)–(*iii*) together with the condition of broad distributions in all the actual phases. The Rayleigh-distributed field, as well as the additional conditions (*iv*)–(*vi*), will not receive further consideration in this paper. Furthermore, no direct assumption on the statistical independence of the actual phases $\Phi_1, \ldots, \Phi_M$ will be made; that is, all random scatterers will not explicitly be assumed to have uncorrelated positions. Although condition (*iii*) is defined in terms of the primary phases, the independence of the actual phases from each A_k is also not assumed. Reducing these restrictions generalizes the problem so that it may be applicable to the multipath interference considered in certain studies on ducted propagation, where one of the reflecting boundaries is random in shape.[8,12,13] In particular, the individual arrivals at a receiving point may be considered, under proper conditions, to be components of an incoherent field. Its mean intensity would then be given by Eq. (3).

In Section 2, the joint probability density function of the amplitudes $A_1, \ldots, A_M$ and the actual phases $\Phi_1, \ldots, \Phi_M$ of the elementary waves is considered. The Fourier transform of this function is taken with respect to just the phase variables, and a characteristic property of the Fourier spectrum for incoherence is derived. A special case of this property is shown, in Section 3, to yield broad distributions in all the actual phases. In Section 4, the joint and marginal probability distributions for the resultant amplitude A and phase Φ of an incoherent field are determined. A summary follows in Section 5.

2. ANALYSIS OF THE ELEMENTARY WAVES

The joint *probability density function* (pdf) of $(A_1, \Phi_1, A_2, \Phi_2, \ldots, A_M, \Phi_M)$ shall be denoted by $h(a_1, \phi_1, \ldots, a_M, \phi_M)$, while the joint pdf of $(A_1, \hat{\Phi}_1, \ldots, A_M, \hat{\Phi}_M)$ is denoted by $\hat{h}(a_1, \phi_1, \ldots, a_M, \phi_M)$. Without loss of generality, all primary phases $\hat{\Phi}_j$ are assumed to be distributed on the same 2π-rad interval, namely, the interval $(-\pi, \pi)$. Then, it can be shown that[3]

$$\hat{h}(a_1, \phi_1, \ldots, a_M, \phi_M) = \begin{cases} \sum_{k_1=-\infty}^{\infty} \sum_{k_2=-\infty}^{\infty} \cdots \sum_{k_M=-\infty}^{\infty} h(a_1, \phi_1 + 2k_1\pi, \ldots, a_M, \phi_M + 2k_M\pi), \\ \qquad \text{for} \quad -\pi \leqslant \phi_j < \pi, \quad j = 1, \ldots, M, \\ 0, \qquad \text{for any } \phi_j \text{ elsewhere} \end{cases} \tag{6}$$

Therefore, $\hat{h}(a_1, \phi_1, \ldots, a_M, \phi_M)$ can be represented by a Fourier series in the variables ϕ_j, when $-\pi < \phi_j < \pi$, $j = 1, \ldots, M$, and is given by

$$\hat{h}(a_1, \phi_1, \ldots, a_M, \phi_M) = \sum_{n_1=-\infty}^{\infty} \sum_{n_2=-\infty}^{\infty} \cdots \sum_{n_M=-\infty}^{\infty} B_{n_1 n_2 \cdots n_M}(a_1, \ldots, a_M) \exp\left(-i \sum_{j=1}^{M} n_j \phi_j\right) \tag{7}$$

With the aid of Eq. (6), the coefficients $B_{n_1 \cdots n_M}(a_1, ..., a_M)$ are observed to be

$$\begin{aligned}
&B_{n_1 \cdots n_M}(a_1, ..., a_M) \\
&= (2\pi)^{-M} \int_{-\pi}^{\pi} \int_{-\pi}^{\pi} \cdots \int_{-\pi}^{\pi} \hat{h}(a_1, \phi_1, ..., a_M, \phi_M) \exp\left(i \sum_{j=1}^{M} n_j \phi_j\right) d\phi_1 \, d\phi_2 \cdots d\phi_M \\
&= (2\pi)^{-M} \sum_{k_1=-\infty}^{\infty} \cdots \sum_{k_M=-\infty}^{\infty} \int_{-\pi}^{\pi} \cdots \int_{-\pi}^{\pi} h(a_1, \phi_1 + 2k_1\pi, ..., a_M, \phi_M + 2k_M\pi) \\
&\quad \times \exp\left[i \sum_{j=1}^{M} n_j(\phi_j + 2k_j\pi)\right] d\phi_1 \cdots d\phi_M
\end{aligned} \tag{8}$$

where it is noted that

$$\exp\left[i \sum_{j=1}^{M} n_j(\phi_j + 2k_j\pi)\right] = \exp\left(i \sum_{j=1}^{M} n_j \phi_j\right)$$

Therefore, from Eq. (8), it can be shown that

$$\begin{aligned}
&B_{n_1 \cdots n_M}(a_1, ..., a_M) \\
&\quad = (2\pi)^{-M} \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} h(a_1, \theta_1, ..., a_M, \theta_M) \exp\left(i \sum_{j=1}^{M} n_j \theta_j\right) d\theta_1 \cdots d\theta_M
\end{aligned} \tag{9}$$

The Fourier transform of $h(a_1, \phi_1, ..., a_M, \phi_M)$ with respect to the variables ϕ_j is given by

$$\begin{aligned}
&C_h(\nu_1, ..., \nu_M; a_1, ..., a_M) \\
&\quad = \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} h(a_1, \phi_1, ..., a_M, \phi_M) \exp\left(i \sum_{j=1}^{M} \nu_j \phi_j\right) d\phi_1 \cdots d\phi_M
\end{aligned} \tag{10}$$

Therefore, by comparing Eqs. (9) and (10), the Fourier coefficients in Eq. (7) are $(2\pi)^{-M}$ times the Fourier transform of h (with respect to the variables ϕ_j) evaluated at the integer quantities $n_1, n_2, ..., n_M$. It is observed from Eq. (10) that

$$\begin{aligned}
C_h(0, 0, ..., 0; a_1, ..., a_M) &= \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} h(a_1, \phi_1, ..., a_M, \phi_M) \, d\phi_1 \cdots d\phi_M \\
&= g(a_1, ..., a_M)
\end{aligned} \tag{11}$$

where $g(a_1, ..., a_M)$ denotes the marginal pdf of $(A_1, ..., A_M)$. Therefore, from Eqs. (6), (7), and (9)–(11), it can be shown that

$$\hat{h}(a_1, \phi_1, ..., a_M, \phi_M) = \left[\prod_{j=1}^{M} u(\phi_j)\right] [g(a_1, ..., a_M) + \epsilon_h(a_1, \phi_1, ..., a_M, \phi_M)] \tag{12}$$

where $u(x)$ is the pdf of a random variable uniformly distributed on the interval $(-\pi, \pi)$, so that

$$u(x) = \begin{cases} (2\pi)^{-1}, & \text{for } -\pi \leqslant x < \pi \\ 0, & \text{for } x \text{ elsewhere} \end{cases} \tag{13}$$

and ϵ_h denotes the function whose Fourier expansion is

$$\epsilon_h(a_1, \phi_1, ..., a_M, \phi_M)$$
$$= \left(\sum_{n_1=-\infty}^{\infty} \cdots \sum_{n_M=-\infty}^{\infty}\right)' C_h(n_1, ..., n_M; a_1, ..., a_M) \exp\left(-i \sum_{j=1}^{M} n_j \phi_j\right) \tag{14}$$

The prime immediately following the $\sum$'s in Eq. (14) is inserted to indicate that the term for which $n_1 = n_2 = \cdots = n_M = 0$ is omitted from the summation. From Eq. (12), conditions (*i*)–(*iii*) for incoherence are observed to be satisfied if and only if $\epsilon_h \equiv 0$ a.e.[2]

Applying Parseval's equation to the Fourier series in Eq. (14), we obtain

$$\int_{-\pi}^{\pi} \cdots \int_{-\pi}^{\pi} |(2\pi)^{-M} \epsilon_h(a_1, \phi_1, ..., a_M, \phi_M)|^2 \, d\phi_1 \cdots d\phi_M$$
$$= (2\pi)^{-M} \left(\sum_{n_1=-\infty}^{\infty} \cdots \sum_{n_M=-\infty}^{\infty}\right)' | C_h(n_1, ..., n_M; a_1, ..., a_M)|^2 \tag{15}$$

From Eq. (15), $\epsilon_h \equiv 0$ a.e. if and only if the Fourier coefficients in Eq. (14) vanish. Therefore, the scattered-wave field, given by Eq. (2), is incoherent if and only if

$$C_h(n_1, ..., n_M; a_1, ..., a_M) \equiv 0 \tag{16}$$

for all integer values of $n_1, ..., n_M$, not all of which are zero, and for all real values of $a_1, ..., a_M$, except possibly for a collection of vectors $(a_1, ..., a_M)$ of measure zero. Then, from Eq. (10), incoherence is equivalent to requiring the Fourier components of $h(a_1, \phi_1, ..., a_M, \phi_M)$, as a function of $\phi_1, ..., \phi_M$, to have zero spectral densities whenever their angular frequency vectors $(\nu_1, ..., \nu_M)$ differ from the zero vector $(0, 0, ..., 0)$ and have all integer components.

An example of h for which $C_h \neq 0$ at noninteger values of $\nu_1, ..., \nu_M$ is given by

$$h(a_1, \phi_1, ..., a_M, \phi_M) = \prod_{j=1}^{M} \tilde{u}(\phi_j) \, g(a_1, ..., a_M) \tag{17}$$

where $\tilde{u}(x)$ is the pdf of a random variable uniformly distributed on the interval $(q, q + 2k\pi)$. The quantity k is taken to be some positive integer, and q is some real number. Using Eq. (10) directly, C_h is shown to satisfy Eq. (16). However, from Eq. (17), the actual phases are observed to be independent, as are the random vectors $(A_1, ..., A_M)$ and $(\Phi_1, ..., \Phi_M)$. Therefore, the policy stated in Section 1 is contradicted. Furthermore, the condition requiring broadly distributed actual phases possesses no significance here. In the following section, a more satisfactory class of pdf's h is given, which is consistent with the motivation introduced in Section 1.

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[2] The statement $\epsilon_h \equiv 0$ a.e., where a.e. is the abbreviation for "almost everywhere," means that the collection of all points $(a_1, \phi_1, ..., a_M, \phi_M)$ at which $\epsilon_h \neq 0$ form an event of zero probability (i.e., zero *measure*). See, for example, ref. 16.

3. THE BAND-LIMITED CASE

In this paper, the pdf $h(a_1, \phi_1, ..., a_M, \phi_M)$ is defined to be *band-limited* (with respect to the variables $\phi_1, ..., \phi_M$) if, for some positive number b, $C_h(\nu_1, ..., \nu_M; a_1, ..., a_M) = 0$ a.e. for all real values of a_j whenever the ν_j are real-valued and at least one ν_k satisfies $|\nu_k| > b$. If b is the smallest such number for which this condition holds, then b is called the *bandwidth* of h (with respect to the variables $\phi_1, ..., \phi_M$).

If the bandwidth b is smaller than unity, then Eq. (16) is valid for all integer values of $n_1, ..., n_M$, not all of which are zero. Therefore, the scattered field is observed to be incoherent if h is band-limited with bandwidth b, and if $b < 1$.

Denoting the marginal pdf of the actual phase Φ_j by $p_j(\phi)$, $1 \leqslant j \leqslant M$, and letting $C_{pj}(\nu)$ be its characteristic function, it can be shown that

$$C_{pj}(\nu) = \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} C_h(\nu\delta_{1j}, \nu\delta_{2j}, ..., \nu\delta_{Mj}; a_1, ..., a_M)\, da_1 \cdots da_M \tag{18}$$

where δ_{kj} denotes the Kronecker delta, so that $\delta_{kj} = 0$ for $k \neq j$, and $\delta_{jj} = 1$. Therefore, if h is band-limited with bandwidth b, then p_j is band-limited with some bandwidth b_j. That is, $C_{pj}(\nu) = 0$ whenever ν is real-valued and $|\nu| > b_j$; furthermore, b_j is the smallest positive number for which this condition holds. It is observed that

$$b_j \leqslant b, \qquad 1 \leqslant j \leqslant M \tag{19}$$

Therefore, if h is band-limited, then $p_j(\phi)$ cannot vanish identically outside a finite interval.[10]

A band-limited h, then, implies, in a qualitative sense, broadly distributed actual phases Φ_j, $1 \leqslant j \leqslant M$. In order to quantitatively investigate this problem, the uncertainty principle[11] is used.

For each actual phase Φ_j, two other random variables shall be defined. The *modified-Φ_j*, denoted by $\Phi_j^{[m]}$, is the random variable whose pdf is given by

$$p_j^{[m]}(\phi) = [p_j(\phi)]^2 \Big/ \int_{-\infty}^{\infty} [p_j(t)]^2\, dt, \qquad 1 \leqslant j \leqslant M \tag{20}$$

and the *spectral-Φ_j*, denoted by $\Phi_j^{[s]}$, is the random variable whose pdf is given by

$$p_j^{[s]}(\nu) = |C_{pj}(\nu)|^2 \Big/ \int_{-\infty}^{\infty} |C_{pj}(t)|^2\, dt, \qquad 1 \leqslant j \leqslant M \tag{21}$$

In these terms, the uncertainty principle[11] states that

$$\sigma(\Phi_j^{[m]})\, \sigma(\Phi_j^{[s]}) \geqslant \tfrac{1}{2}, \qquad 1 \leqslant j \leqslant M \tag{22}$$

where σ denotes standard deviation. Therefore, the more narrow the Fourier spectrum of $p_j(\phi)$, the broader will be the distribution of Φ_j; and the more narrow the distribution of Φ_j, the broader will be the Fourier spectrum of $p_j(\phi)$.

Although the dispersion of the outcomes of Φ_j about its mean value is usually measured by $\sigma(\Phi_j)$, Eq. (22) requires that this dispersion be measured by $\sigma(\Phi_j^{[m]})$. This assumes that the shape of the graph of $[p_j(\phi)]^2$ is not too different from that of $p_j(\phi)$. Similarly, the dispersion of the Fourier spectrum of $p_j(\phi)$ about the spectrum's central moment ought to be measured by the standard deviation of the random variable $\tilde{\Phi}_j^{[s]}$, whose pdf is given by

$$\tilde{p}_j^{[s]}(\nu) = | C_{pj}(\nu)| \Big/ \int_{-\infty}^{\infty} | C_{pj}(t)| \, dt \tag{23}$$

However, if the graphs of $| C_{pj}(\nu)|$ and $| C_{pj}(\nu)|^2$ are similar in shape, then $\sigma(\Phi_j^{[s]})$ may be used as reliably as $\sigma(\tilde{\Phi}_j^{[s]})$.

It is observed that the magnitude of the characteristic function $C_{pj}(\nu)$ is an even function of ν. Therefore, from Eq. (21), it can be shown that

$$\sigma(\Phi_j^{[s]}) = \{E[(\Phi_j^{[s]})^2]\}^{1/2}, \qquad 1 \leqslant j \leqslant M \tag{24}$$

where E denotes mathematical expectation. Substituting Eq. (24) into Eq. (22), we obtain

$$\sigma(\Phi_j^{[m]})\{E[(\Phi_j^{[s]})^2]\}^{1/2} \geqslant \tfrac{1}{2}, \qquad 1 \leqslant j \leqslant M \tag{25}$$

Since $p_j(\phi)$ is band-limited with bandwidth b_j, then, from Eq. (21), $p_j^{[s]}(\nu) = 0$ whenever $| \nu | > b_j$. Therefore, using Eq. (19), it is observed that

$$E[(\Phi_j^{[s]})^2] = \int_{-b_j}^{b_j} \nu^2 p_j^{[s]}(\nu) \, d\nu \leqslant b_j^{\,2} \int_{-b_j}^{b_j} p_j^{[s]}(\nu) \, d\nu = b_j^{\,2} \leqslant b^2 \tag{26}$$

From Eqs. (25) and (26), we obtain

$$\sigma(\Phi_j^{[m]}) \geqslant \tfrac{1}{2} b^{-1}, \qquad 1 \leqslant j \leqslant M \tag{27}$$

Therefore, Eq. (27) exhibits a lower bound on the standard deviation of each modified-Φ_j, $1 \leqslant j \leqslant M$, in terms of the bandwidth of a band-limited pdf h. It is observed to decrease monotonically with increasing b and equal the value $\frac{1}{2}$ rad when $b = 1$.

In this section, a band-limited pdf h has been shown to imply an incoherently scattered field if its bandwidth b is smaller than unity. Furthermore, the actual phases have been found to be broadly distributed random variables, and the standard deviations of the modified-Φ_j are all greater than or equal to $(2b)^{-1}$.
Neither has the statistical independence among the actual phases been explicitly assumed, nor has that between the actual phases Φ_j and the amplitudes A_k. However, it should be mentioned that whether these conditions are implied by a band-limited h remains a theoretical question, and is open to further investigation.

4. THE TOTAL SCATTERED FIELD

In this section, the joint pdf of (A, Φ), denoted by $t(a, \phi)$, is consider. The function $t(a, \phi)$ is derived in terms of the joint probability distribution of $(A_1, \Phi_1, \ldots, A_M, \Phi_M)$, and is presented in the form

$$t(a, \phi) = u(\phi)[g_T(a) + \delta_h(a, \phi)] \tag{28}$$

where $g_T(a)$ is determined from g. The function $u(\phi)$ is given by Eq. (13), and the function $\delta_h(a, \phi)$ is determined from ϵ_h later in the section.

4.1. Mathematical Analysis

If the rectangular-coordinate components of $A_j \exp(i\Phi_j)$, $j = 1,..., M$ and $Ae^{i\Phi}$ are defined by

$$X_j = A_j \cos \Phi_j , \qquad Y_j = A_j \sin \Phi_j , \qquad 1 \leqslant j \leqslant M \tag{29}$$

and

$$X = A \cos \Phi, \qquad Y = A \sin \Phi \tag{30}$$

then Eq. (2) is equivalent to

$$X = \sum_{j=1}^{M} X_j , \qquad Y = \sum_{j=1}^{M} Y_j \tag{31}$$

The joint pdf of the random vector $(X_1 , Y_1 ,..., X_M , Y_M)$ is denoted by $r(x_1 , y_1 ,..., x_M , y_M)$. Using Eq. (31) as a transformation of the random vector $(X_1 , Y_1 ,..., X_M , Y_M)$ to the vector $(X_1 , Y_1 ,..., X_{M-1} , Y_{M-1} , X, Y)$, the joint pdf of the second vector is denoted by $s(x_1 , y_1 ,..., x_{M-1} , y_{M-1} , x, y)$, and the Jacobian of this transformation can be shown to equal unity. Then,

$$\begin{aligned} &s(x_1 , y_1 ,..., x_{M-1} , y_{M-1} , x, y) \\ &\quad = r\left(x_1 , y_1 ,..., x_{M-1} , y_{M-1} , \quad x - \sum_{j=1}^{M-1} x_j , \quad y - \sum_{j=1}^{M-1} y_j\right) \end{aligned} \tag{32}$$

Therefore, using Eq. (32), the joint pdf $q(x, y)$ of (X, Y), which is a marginal distribution of $(X_1 , Y_1 ,..., X_{M-1} , Y_{M-1} , X, Y)$, is given by

$$\begin{aligned} q(x, y) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} r\left(x_1 , y_1 ,..., x_{M-1} , y_{M-1} , \quad x - \sum_{j=1}^{M-1} x_j , \quad y - \sum_{j=1}^{M-1} y_j\right) \\ \times\, dx_1\, dy_1 \cdots dx_{M-1}\, dy_{M-1} \end{aligned} \tag{33}$$

Using Eq. (29) to transform Eq. (12) into rectangular coordinates, we obtain

$$\begin{aligned} &r(x_1 , y_1 ,..., x_M , y_M) \\ &\quad = (2\pi)^{-M} \left[\prod_{j=1}^{M} (x_j^2 + y_j^2)^{-1/2}\right] \{g[(x_1^2 + y_1^2)^{1/2},..., (x_M^2 + y_M^2)^{1/2}] \\ &\qquad + \epsilon_h[(x_1^2 + y_1^2)^{1/2}, F(x_1 , y_1),..., (x_M^2 + y_M^2)^{1/2}, F(x_M , y_M)]\} \end{aligned} \tag{34}$$

where

$$F(a, b) = \arg(a + ib) \equiv \begin{cases} -\operatorname{Arccos}[a(a^2 + b^2)^{-1/2}], & \text{for } \ b < 0 \\ \ \ \operatorname{Arccos}[a(a^2 + b^2)^{-1/2}], & \text{for } \ b \geqslant 0 \end{cases} \tag{35}$$

and $\prod_{j=1}^{M} (x_j^2 + y_j^2)^{-1/2}$ is the Jacobian of the transformation. Substituting Eq. (34) into Eq. (33) and changing the variables of integration to those of polar coordinates, we obtain, after simplifying,

$$q(x, y) = (2\pi)^{-M} \int_{\phi_{M-1}=-\pi}^{\pi} \int_{a_{M-1}=0}^{\infty} \cdots \int_{\phi_1=-\pi}^{\pi} \int_{a_1=0}^{\infty}$$

$$\times \left[\left(x - \sum_{j=1}^{M-1} a_j \cos \phi_j\right)^2 + \left(y - \sum_{j=1}^{M-1} a_j \sin \phi_j\right)^2\right]^{-1/2}$$

$$\times \left\{g \left\{a_1 ,..., a_{M-1} , \left[\left(x - \sum_{j=1}^{M-1} a_j \cos \phi_j\right)^2 + \left(y - \sum_{j=1}^{M-1} a_j \sin \phi_j\right)^2\right]^{1/2}\right\}\right.$$

$$+ \epsilon_h \left\{a_1, \phi_1,..., a_{M-1}, \phi_{M-1}, \left[\left(x - \sum_{j=1}^{M-1} a_j \cos \phi_j\right)^2 + \left(y - \sum_{j=1}^{M-1} a_j \sin \phi_j\right)^2\right]^{1/2},\right.$$

$$\left.\left. F\left(x - \sum_{j=1}^{M-1} a_j \cos \phi_j , y - \sum_{j=1}^{M-1} a_j \sin \phi_j\right)\right\}\right\} da_1\, d\phi_1 \cdots da_{M-1}\, d\phi_{M-1} \quad (36)$$

Recalling that the joint pdf of (A, Φ) is denoted by $t(a, \phi)$, and using Eq. (30) to transform (X, Y) to (A, Φ), we obtain

$$t(a, \phi) = \begin{cases} aq(a \cos \phi, a \sin \phi), & \text{for } a \geqslant 0 \text{ and } -\pi \leqslant \phi < \pi \\ 0, & \text{for } a \text{ or } \phi \text{ elsewhere} \end{cases} \quad (37)$$

Noting the identity,

$$\left(a \cos \phi - \sum_{j=1}^{M-1} a_j \cos \phi_j\right)^2 + \left(a \sin \phi - \sum_{j=1}^{M-1} a_j \sin \phi_j\right)^2$$

$$= a^2 - 2a \sum_{j=1}^{M-1} a_j \cos(\phi_j - \phi)$$

$$+ \sum_{j=1}^{M-1} \sum_{k=1}^{M-1} a_j a_k [\cos(\phi_j - \phi) \cos(\phi_k - \phi) + \sin(\phi_j - \phi) \sin(\phi_k - \phi)] \quad (38)$$

Eqs. (36) and (38) yield, for $a \geqslant 0$ and $-\pi \leqslant \phi < \pi$,

$$q(a \cos \phi, a \sin \phi)$$

$$= (2\pi)^{-M} \int_{-\pi}^{\pi} \int_{0}^{\infty} \cdots \int_{-\pi}^{\pi} \int_{0}^{\infty} f_g[a, a_1 ,..., a_{M-1} ,$$

$$\cos(\phi_1 - \phi),..., \cos(\phi_{M-1} - \phi), \sin(\phi_1 - \phi),..., \sin(\phi_{M-1} - \phi)]$$

$$\times da_1\, d\phi_1 \cdots da_{M-1}\, d\phi_{M-1} + (2\pi)^{-1} a^{-1} \delta_h(a, \phi) \quad (39)$$

where

$$\begin{aligned} f_g(a, a_1, \ldots, a_{M-1}, \xi_1, \ldots, \xi_{M-1}, \eta_1, \ldots, \eta_{M-1}) \\ = [\mathcal{A}_M(a, a_1, \ldots, a_{M-1}, \xi_1, \ldots, \xi_{M-1}, \eta_1, \ldots, \eta_{M-1}]^{-1} \\ \times g[a_1, \ldots, a_{M-1}, \mathcal{A}_M(a, a_1, \ldots, a_{M-1}, \xi_1, \ldots, \xi_{M-1}, \eta_1, \ldots, \eta_{M-1})] \end{aligned} \tag{40}$$

and

$$\begin{aligned} \mathcal{A}_M(a, a_1, \ldots, a_{M-1}, \xi_1, \ldots, \xi_{M-1}, \eta_1, \ldots, \eta_{M-1}) \\ = \left[a^2 - 2a \sum_{j=1}^{M-1} a_j \xi_j + \sum_{j=1}^{M-1} \sum_{k=1}^{M-1} a_j a_k (\xi_j \xi_k + \eta_j \eta_k)\right]^{1/2} \end{aligned} \tag{41}$$

The quantity $\delta_h(a, \phi)$ in Eq. (39) is given by

$$\delta_h(a, \phi) = \begin{cases} (2\pi)^{-(M-1)} a \int_{-\pi}^{\pi} \int_0^{\infty} \cdots \int_{-\pi}^{\pi} \int_0^{\infty} f_{\epsilon h}(a, \phi, a\ \ , \phi_1, \ldots, a_{M-1}, \phi_{M-1}) \\ \quad \times da_1\, d\phi_1 \cdots da_{M-1}\, d\phi_{M-1}, & \text{for} \quad a \geqslant 0 \\ 0, & \text{for} \quad a < 0 \end{cases} \tag{42}$$

where

$$\begin{aligned} & f_{\epsilon h}(a, \phi, a_1, \phi_1, \ldots, a_{M-1}, \phi_{M-1}) \\ & = \{\mathcal{A}_M[a, a_1, \ldots, a_{M-1}, \cos(\phi_1 - \phi), \ldots, \cos(\phi_{M-1} - \phi), \sin(\phi_1 - \phi), \ldots, \sin(\phi_{M-1} - \phi)]\}^{-1} \\ & \quad \times \epsilon_h \Bigg\{ a_1, \phi_1, \ldots, a_{M-1}, \phi_{M-1}, \\ & \quad \left[\left(a \cos\phi - \sum_{j=1}^{M-1} a_j \cos\phi_j\right)^2 + \left(a \sin\phi - \sum_{j=1}^{M-1} a_j \sin\phi_j\right)^2\right]^{1/2}, \\ & \quad F\left(a \cos\phi - \sum_{j=1}^{M-1} a_j \cos\phi_j, a \sin\phi - \sum_{j=1}^{M-1} a_j \sin\phi_j\right)\Bigg\} \end{aligned} \tag{43}$$

Letting $\psi_j = \phi_j - \phi$, $1 \leqslant j \leqslant M - 1$, in the integral in Eq. (39), we obtain, for $a \geqslant 0$ and $-\pi \leqslant \phi < \pi$,

$$\begin{aligned} & q(a \cos\phi, a \sin\phi) \\ & = (2\pi)^{-M} \int_{-\pi-\phi}^{\pi-\phi} \int_0^{\infty} \cdots \int_{-\pi-\phi}^{\pi-\phi} \int_0^{\infty} f_g(a, a_1, \ldots, a_{M-1}, \cos\psi_1, \ldots, \cos\psi_{M-1}, \\ & \quad \sin\psi_1, \ldots, \sin\psi_{M-1})\, da_1\, d\psi_1 \cdots da_{M-1}\, d\psi_{M-1} + (2\pi)^{-1} a^{-1} \delta_h(a, \phi) \end{aligned} \tag{44}$$

Since the integrand in Eq. (44) is periodic in each ψ_j with period 2π rad, and since each ψ_j-interval of integration, namely $(-\pi - \phi, \pi - \phi)$, is always of length

2π rad, then the integral in Eq. (44) is independent of ϕ. Therefore, from Eqs. (37) and (44), we obtain Eq. (28), where $g_T(a)$ is given by

$$g_T(a) = \begin{cases} (2\pi)^{-(M-1)} a \int_{-\pi}^{\pi} \int_0^{\infty} \cdots \int_{-\pi}^{\pi} \int_0^{\infty} f_g(a, a_1, ..., a_{M-1}, \cos\psi_1, ..., \cos\psi_{M-1}, \\ \qquad \sin\psi_1, ..., \sin\psi_{M-1})\, da_1\, d\psi_1 \cdots da_{M-1}\, d\psi_{M-1}, & \text{for} \quad a \geqslant 0 \\ 0, & \text{for} \quad a < 0 \end{cases} \tag{45}$$

As claimed earlier, δ_h is determined from ϵ_h, and is given by Eqs. (41)–(43). Also, g_T is determined from g, and is given by Eqs. (40), (41), and (45).

If $\epsilon_h \equiv 0$, then Eqs. (42) and (43) show that $\delta_h \equiv 0$. Since $\epsilon_h \equiv 0$ is equivalent to incoherence, therefore Eqs. (13) and (28) show that the resultant phase of an incoherent field is independent of the resultant amplitude and is uniformly distributed on the interval $(-\pi, \pi)$. The marginal pdf of the resultant amplitude is given by Eq. (45).

From Eqs. (2), (38), and (41), it can be shown that

$$\begin{aligned} A_M = \mathcal{A}_M[&A, A_1, ..., A_{M-1}, \cos(\Phi_1 - \Phi), ..., \cos(\Phi_{M-1} - \Phi), \\ &\sin(\Phi_1 - \Phi), ..., \sin(\Phi_{M-1} - \Phi)] \end{aligned} \tag{46}$$

Therefore, the function $\mathcal{A}_M$ gives the amplitude of the Mth elementary vector, $A_M \exp(i\Phi_M)$, in terms of the resultant of all M elementary vectors, $Ae^{i\Phi}$, and of the first $M - 1$ elementary vectors, $A_j \exp(i\Phi_j)$, $1 \leqslant j \leqslant M - 1$.

The presence of $\mathcal{A}_M$ in the integrals in Eqs. (39), (42), (44), and (45) is due to the product of two Jacobians. The first Jacobian results from the transformation that yields Eq. (34), and the second is needed in Eq. (36), where the variables of integration were changed from those of rectangular coordinates to those of polar coordinates.

Since the random variables A_j, $1 \leqslant j \leqslant M$, and A never assume negative outcomes, the functions h, $\hat{h}$, g, and ϵ_h vanish if any a_j is negative. Therefore, the interval of integration for the variables a_j, $1 \leqslant j \leqslant M - 1$, in Eqs. (36), (39), (42), and (45) is taken to be $(0, \infty)$ instead of $(-\infty, \infty)$. Also, the joint pdf $t(a, \phi)$, as well as the functions g_T and δ_h, are defined to vanish if $a < 0$.

4.2. Remarks

It should be mentioned that the absence of the random amplitudes A_j from consideration reduces the problem treated in this paper to one that was previously investigated in the literature,[14,15] though not on propagation. However, from the definitions of incoherence, the amplitudes A_j of the elementary waves must also be considered: Condition (*iii*) of Section 1 is needed to demonstrate Eq. (3) for random A_j as well as to derive the independence of A from Φ.

5. CONCLUSION

This paper defines the incoherence of a scattered field in terms of the amplitudes and primary phases of the elementary waves. However, conditions involving the actual phases, rather than the primary phases, are derived for incoherence.

The joint probability density function of the random variables A_1, Φ_1,..., A_M, Φ_M, denoted by $h(a_1, \phi_1, ..., a_M, \phi_M)$, is treated by taking its Fourier transform, denoted by $C_h(\nu_1, ..., \nu_M; a_1, ..., a_M)$, with respect to only the variables $\phi_1, ..., \phi_M$. The scattered field is shown to be incoherent if and only if $C_h(n_1, ..., n_M; a_1, ..., a_M)$ vanishes, independently of the a_j, for all integer values of $n_1, ..., n_M$, not all of which are zero. Therefore, if h is band-limited, so that C_h vanishes whenever any ν_j exceeds, in magnitude, the bandwidth b, and if b smaller than unity, then the scattered field is incoherent. Furthermore, by using the uncertainty principle, a band-limited h is shown to imply broad distributions for all the random phases $\Phi_1, ..., \Phi_M$.

The resultant phase Φ of an incoherently scattered field is shown to be independent of the resultant amplitude A and uniformly distributed on the interval $(-\pi, \pi)$. These properties of A and Φ, which are also valid for a Rayleigh-distributed field resulting from the addition of conditions (*iv*)–(*vi*), are therefore shown to require only the conditions (*i*)–(*iii*) for incoherence. Furthermore, under the conditions of incoherence, the probability density function of A is derived in terms of the joint probability density function of $A_1, ..., A_M$.

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